

VARIABLE-BASIS FUZZY FILTERS

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ABSTRACT. U. Höhle and A. Šostak have developed in [3] the category of complete quasi-monoidal lattices; S. E. Rodabaugh in [6] proposed its opposite category and with a subcategory \mathbf{C} of the latter, he define grounds of the form $\mathbf{SET} \times \mathbf{C}$.

In this paper, for each ground category of the form $\mathbf{SET} \times \mathbf{C}$, we study categorical frameworks for variable-basis fuzzy filters, particularly the category $\mathbf{C} - \mathbf{FFIL}$ of variable-basis fuzzy filters, as a natural generalization of the category of fixed-basis fuzzy filters which was introduced in [5]. In addition, we get some relations between the category of variable-basis fuzzy filters and the category of variable-basis fuzzy topological spaces.

0. INTRODUCTION

The mathematical theories being created in fuzzy sets are determined by certain data, including the following: the ground (or base) category for a set theory, the powerset operators of the ground category, the forgetful (or underlying) functor from the concrete category into the ground category, and the particular structure carried by the objects of the concrete category and preserved by the morphisms of that category (cf.[1] and [6]).

Variable-basis fuzzy filter (as variable-basis fuzzy topology in [6]) request grounds which are products of two categories: The category \mathbf{SET} , which carries “point-set” information, and the second category, which in this paper will be a subcategory \mathbf{C} of the category $\mathbf{LOQML} \equiv \mathbf{CQML}^{\text{op}}$. The category \mathbf{CQML} of complete quasi-monoidal lattices was being developed

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by U. Höhle and A. Šostak in [3].

Following P. T. Johnstone ([4]), within the text of the paper, those propositions, lemmas and theorems whose proofs require Zorn's lemma are distinguished by being marked with an asterisk.

1. GROUND CATEGORIES $\mathbf{SET} \times \mathbf{C}$ AND $\mathbf{SET} \times \mathbf{L}_\Phi$

1.1. Ground categories. S. E. Rodabaugh presents in [6] the ground categories $\mathbf{SET} \times \mathbf{C}$ and $\mathbf{SET} \times \mathbf{L}_\Phi$ as follows:

Definition 1.1. [6] *Let \mathbf{C} be a subcategory of \mathbf{LOQML} . The ground category $\mathbf{SET} \times \mathbf{C}$ comprises the following data:*

- (SC1) **Objects.** (X, L) , where $X \in |\mathbf{SET}|$ and $L \in |\mathbf{C}|$. The object (X, L) is a **(ground) set**.
- (SC2) **Morphisms.** $(f, \Phi) : (X, L) \rightarrow (Y, M)$, where $f : X \rightarrow Y$ in \mathbf{SET} , $\Phi : L \rightarrow M$ in \mathbf{C} . The morphism (f, Φ) is a **(ground) function**.
- (SC3) **Composition.** Component-wise in the respective categories.
- (SC4) **Identities.** Component-wise in the respective categories, i.e. $id_{(X, L)} = (id_X, id_L)$.

Definition 1.2. [6] *Let \mathbf{C} be a subcategory of \mathbf{LOQML} , $L \in |\mathbf{C}|$ and $\Phi \in \text{Hom}(\mathbf{C}, \mathbf{C})$. The ground category $\mathbf{SET} \times \mathbf{L}_\Phi$ comprises the following data:*

- (SL1) **Objects.** (X, L) , where $X \in |\mathbf{SET}|$. The object (X, L) is a **(ground) set**.
- (SL2) **Morphisms.** $(f, \Phi) : (X, L) \rightarrow (Y, L)$, where $f : X \rightarrow Y$ in \mathbf{SET} . The morphism (f, Φ) is a **(ground) function**.
- (SL3) **Composition.** $(f, \Phi) \circ (g, \Phi) = (f \circ g, \Phi)$, where the composition in the first component is in \mathbf{SET} .

(SL4) **Identities.** $id_{(X,L)} = (id_X, \Phi)$, where the identity in the first component is in **SET**.

Remark 1.3 (Powerset operators). For $(f, \Phi) : (X, L) \rightarrow (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$ or $\mathbf{SET} \times \mathbf{L}_\Phi$, the forward powerset operator, $(f, \Phi)^\rightarrow : L^X \rightarrow M^Y$ is defined by

$$(f, \Phi)^\rightarrow(a) = \bigwedge \{b \mid f_L^\rightarrow(a) \leq (\Phi^{op})(b)\},$$

where $f_L^\rightarrow : L^X \rightarrow L^Y$ is defined by $f_L^\rightarrow(a)(y) = \bigvee_{f(x)=y} a(x)$, and $(\Phi^{op}) : L^Y \leftarrow M^Y$ is defined by $(\Phi^{op})(b) = \Phi^{op} \circ b$. The backward powerset operator $(f, \Phi)^\leftarrow : L^X \leftarrow M^Y$ is defined by

$$(f, \Phi)^\leftarrow(b) = \Phi^{op} \circ b \circ f,$$

in other words, that diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ (f, \Phi)^\leftarrow(b) \downarrow & & \downarrow b \\ L & \xleftarrow{\Phi^{op}} & M \end{array}$$

is commutative.

Remark 1.4.

- (1) Note that $\mathbf{SET} \times \mathbf{L}_\Phi$ is a subcategory of $\mathbf{SET} \times \mathbf{C}$ if and only if $\Phi = id$.
- (2) When Φ^{op} preserves arbitrary meets, it has a co-adjoint (or left adjoint):

$$*\Phi : L \rightarrow M, \quad \text{defined by } *\Phi(\alpha) = \bigwedge \{\beta \in M \mid \alpha \leq \Phi^{op}(\beta)\}$$

- (3) Clearly, $(f, \Phi)^\leftarrow$ and $(f, \Phi)^\rightarrow$ constitute an adjoint pair.
- (4) (f, Φ) is an isomorphism in $\mathbf{SET} \times \mathbf{C}$ iff f and Φ are isomorphisms in **SET** and **C**, respectively, iff f and Φ are bijections.

2. THE CATEGORY $\mathbf{C} - \mathbf{FFIL}$

The categorical frameworks detailed below rest on a ground category of the form $\mathbf{SET} \times \mathbf{C}$, where \mathbf{C} is a subcategory of \mathbf{LOQML} . This section generalizes the category $\mathbf{C} - \mathbf{FIL}$ studied in [5] for this kind of ground categories.

Definition 2.1. *Let \mathbf{C} be a subcategory of \mathbf{LOQML} . The category $\mathbf{C} - \mathbf{FFIL}$ consists of the following data:*

(FF1) **Objects:** *Ordered triples (X, L, F) satisfying the following axioms:*

- (a) $(X, L) \in |\mathbf{SET} \times \mathbf{C}|$.
- (b) $F : L^X \rightarrow L$ is a mapping, called a **fuzzy filter**, satisfying:
 - (i) $F(1_X) = \top$, where $1_X : X \rightarrow L$ is the constant mapping defined by $1_X(t) = \top$, for all $t \in X$.
 - (ii) $F(0_X) = \perp$, where $0_X : X \rightarrow L$ is the constant mapping defined by $1_X(t) = \perp$, for all $t \in X$.
 - (iii) $f \leq g$ implies $F(f) \leq F(g)$, for all $f, g \in L^X$
 - (iv) $F(f) \otimes F(g) \leq F(f \otimes g)$, for all $f, g \in L^X$
- (c) **Equality of objects:** $(X, L, F) = (Y, M, \Omega)$ if and only if $(X, L) = (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$, and $F = \Omega$ as \mathbf{SET} -mappings from $L^X \equiv M^Y$ to $L \equiv M$.

(FF2) **Morphisms:** *Ordered pairs $(f, \phi) : (X, L, F) \rightarrow (Y, M, \Omega)$, called **fuzzy filter continuous morphisms**, satisfying the following axioms:*

- (a) $(f, \phi) : (X, L) \rightarrow (Y, M)$ is a morphism in $\mathbf{SET} \times \mathbf{C}$
- (b) $\phi^{op} \circ \Omega \leq F \circ (f, \phi)^\leftarrow$ on M^Y .

(FF3) *The composition of morphisms is realized as in the category $\mathbf{SET} \times \mathbf{C}$.*

(X, L, F) is called a **fuzzy filtered set**.

An alternate expression for fuzzy filter continuous morphisms is

Proposition 2.2. *On M^Y the following holds:*

$$\phi^{op} \circ \Omega \leq F \circ (f, \phi)^\leftarrow \quad \text{if and only if} \quad \Omega \leq (\phi^{op})^* \circ F \circ (f, \phi)^\leftarrow,$$

where $(\Phi^{op})^*$ is the right (Galois) adjoint of Φ^{op} .

Proof. For necessity, we apply $(\Phi^{op})^*$ to both sides, invoking the fact that $((\Phi^{op})^*, \Phi^{op})$ is a Galois correspondence; and for sufficiency, we apply Φ^{op} to both sides, invoking again that $((\Phi^{op})^*, \Phi^{op})$ is a Galois correspondence. ■

Proposition 2.3. *Let \mathbf{C} be a subcategory of **LOQML**. Then $\mathbf{C} - \mathbf{FFIL}$ is a concrete category over $\mathbf{SET} \times \mathbf{C}$.*

Proof. Clearly, the forgetful functor $V : \mathbf{C} - \mathbf{FFIL} \rightarrow \mathbf{SET} \times \mathbf{C}$ is faithful. Then, the main point to be checked is composition. Let the following morphisms be given in $\mathbf{C} - \mathbf{FFIL}$:

$$(f, \phi) : (X, L, F_1) \rightarrow (Y, M, F_2) \quad \text{and} \quad (g, \psi) : (Y, M, F_2) \rightarrow (Z, N, F_3),$$

then

$$\begin{aligned} (\psi \circ \phi)^{op} \circ F_3 &= \phi^{op} \circ \psi^{op} \circ F_3 \\ &\leq \phi^{op} \circ F_2 \circ (g, \psi)^\leftarrow \\ &\leq F_1 \circ (f, \phi)^\leftarrow \circ (g, \psi)^\leftarrow \\ &= F_1 \circ (g \circ f, \psi \circ \phi)^\leftarrow. \end{aligned}$$

This shows that $(g \circ f, \psi \circ \phi)$ is a fuzzy filter continuous morphisms. ■

Proposition 2.4. *Let (X, L) be a $\mathbf{SET} \times \mathbf{C}$ -object and let $(X, L, F_\lambda)_{\lambda \in \Lambda}$ be a non-empty family of fuzzy filtered sets. Then the mapping*

$$\mathcal{F} : L^X \rightarrow L \quad \text{defined by} \quad \mathcal{F}(g) = \bigwedge_{\lambda \in \Lambda} F_\lambda(g)$$

is a fuzzy filter on the ground set (X, L) .

Proof. In fact,

- i. $\mathcal{F}(1_X) = \bigwedge_{\lambda \in \Lambda} F_\lambda(1_X) = \bigwedge_{\lambda \in \Lambda} \top = \top$.
- ii. $\mathcal{F}(0_X) = \bigwedge_{\lambda \in \Lambda} F_\lambda(0_X) = \bigwedge_{\lambda \in \Lambda} \perp = \perp$.
- iii. If $f \leq g$ then $F_\lambda(f) \leq F_\lambda(g)$ for each $\lambda \in \Lambda$, therefore

$$\mathcal{F}(f) = \bigwedge_{\lambda \in \Lambda} F_\lambda(f) \leq \bigwedge_{\lambda \in \Lambda} F_\lambda(g) = \mathcal{F}(g).$$

- iv. Finally, for each $f, g \in L^X$ we have

$$\begin{aligned} \mathcal{F}(f) \otimes \mathcal{F}(g) &= \bigwedge_{\lambda \in \Lambda} F_\lambda(f) \otimes \bigwedge_{\lambda \in \Lambda} F_\lambda(g) \leq \bigwedge_{\lambda \in \Lambda} (F_\lambda(f) \otimes F_\lambda(g)) \\ &\leq \bigwedge_{\lambda \in \Lambda} F_\lambda(f \otimes g) = \mathcal{F}(f \otimes g). \end{aligned}$$

■

Theorem 2.5. *Let \mathbf{C} be a subcategory of **LOQML**, let $(f, \Phi) : (X, L) \rightarrow (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$, let $(\Phi^{op})^*$ be the right adjoint of $\Phi^{op} : L \leftarrow M$ and let (X, L, F) be a fuzzy filtered set. Then the following holds:*

- (1) $F_{(f, \Phi)} := (\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow : M^Y \rightarrow M$ is a fuzzy filter,
- (2) $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_{(f, \Phi)})$ is a fuzzy filter continuous morphisms,
- (3) $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_1)$ is a fuzzy filter continuous morphism if and only if $F_1 \leq F_{(f, \Phi)}$,
- (4) $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_{(f, \Phi)})$ is a final morphism in $\mathbf{C} - \mathbf{FFIL}$,
- (5) $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F')$ is a final morphism in $\mathbf{C} - \mathbf{FFIL}$ if and only if $F' \leq F_{(f, \Phi)}$.

Proof. (1) We check the fuzzy filter axioms of $F_{(f, \Phi)}$ as follows:

(a)

$$\begin{aligned}
F_{(f,\Phi)}(1_Y) &= [(\Phi^{op})^* \circ F \circ (f, \Phi)^{\leftarrow}] (1_Y) \\
&= (\Phi^{op})^* (F(\Phi^{op} \circ 1_Y \circ f)) \\
&= (\Phi^{op})^* (F(1_X)), \text{ } (\Phi \text{ is a morphism in } \mathbf{CQML}) \\
&= (\Phi^{op})^*(\top) = \top, \text{ } ((\Phi^{op})^* \text{ is the right adjoint of } \Phi^{op}).
\end{aligned}$$

(b)

$$\begin{aligned}
F_{(f,\Phi)}(0_Y) &= [(\Phi^{op})^* \circ F \circ (f, \Phi)^{\leftarrow}] (0_Y) \\
&= (\Phi^{op})^* (F(\Phi^{op} \circ 0_Y \circ f)) \\
&= (\Phi^{op})^* (F(0_X)) = (\Phi^{op})^*(\perp) = \perp.
\end{aligned}$$

(c) For $h, k \in M^Y$ and $h \leq k$, we have

$$\begin{aligned}
F_{(f,\Phi)}(h) &= [(\Phi^{op})^* \circ F \circ (f, \Phi)^{\leftarrow}] (h) \\
&= (\Phi^{op})^* (F(\Phi^{op} \circ h \circ f)) \\
&\leq (\Phi^{op})^* (F(\Phi^{op} \circ k \circ f)) = F_{(f,\Phi)}(k).
\end{aligned}$$

(d) Let $h, k \in M^Y$ then we have

$$\begin{aligned}
& F_{(f, \Phi)}(h) \otimes F_{(f, \Phi)}(k) \\
&= [(\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow](h) \otimes [(\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow](k) \\
&= (\Phi^{op})^*(F(\Phi^{op} \circ h \circ f)) \otimes (\Phi^{op})^*(F(\Phi^{op} \circ k \circ f)) \\
&= (\Phi^{op})^*[F(\Phi^{op} \circ h \circ f) \otimes F(\Phi^{op} \circ k \circ f)] \\
&\leq (\Phi^{op})^*[F\{(\Phi^{op} \circ h \circ f) \otimes (\Phi^{op} \circ k \circ f)\}] \\
&\leq (\Phi^{op})^*[F\{(\Phi^{op} \circ (h \otimes k) \circ f)\}] \\
&\leq (\Phi^{op})^*[F\{(\Phi^{op} \circ h \circ f) \otimes (\Phi^{op} \circ k \circ f)\}] \\
&= (\Phi^{op})^*[F\{\Phi^{op} \circ (h \otimes k) \circ f\}] \\
&= [(\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow](h \otimes k) \\
&= F_{(f, \Phi)}(h \otimes k)
\end{aligned}$$

(2) Since $\Phi^{op} \circ (\Phi^{op})^* \leq id_L$, we have

$$\begin{aligned}
(\Phi^{op}) \circ F_{(f, \Phi)} &= \Phi^{op} \circ (\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow \\
&\leq F \circ (f, \Phi)^\leftarrow,
\end{aligned}$$

thus $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_{(f, \Phi)})$ is a fuzzy filter continuous morphisms.

(3) Let $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_1)$ be a fuzzy filter continuous morphisms. Then we have

$$\Phi^{op} \circ F_1 \leq F \circ (f, \Phi)^\leftarrow,$$

which implies that

$$\begin{aligned}
F_1 &\leq (\Phi^{op})^* \circ \Phi^{op} \circ F_1 \\
&\leq (\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow \\
&= F_{(f, \Phi)}.
\end{aligned}$$

For necessity, let $F_1 \leq F_{(f,\Phi)}$. Since $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_{(f,\Phi)})$ is a fuzzy filter continuous morphisms, we have

$$\begin{aligned}\Phi^{op} \circ F_1 &\leq \Phi^{op} \circ F_{(f,\Phi)} \\ &\leq F \circ (f, \Phi)^\leftarrow.\end{aligned}$$

(4) To show $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F_{(f,\Phi)})$ is a final morphism in $\mathbf{C} - \mathbf{FFIL}$, we must verify that for each $(Z, N, F') \in \mathbf{C} - \mathbf{FFIL}$, and for each $(g, \Psi) \in \text{Hom}((Y, M), (Z, N))$ in $\mathbf{SET} \times \mathbf{C}$, the following holds:

$$(g, \Psi) \circ (f, \Phi) \in \text{Hom}_{\mathbf{C} - \mathbf{FFIL}}\left((X, L, F), (Z, N, F')\right)$$

implies

$$(g, \Psi) \in \text{Hom}_{\mathbf{C} - \mathbf{FFIL}}\left((Y, M, F_{(f,\Phi)}), (Z, N, F')\right).$$

Since $(g, \Psi) \circ (f, \Phi)$ is a fuzzy filter continuous morphisms, it follows

$$\Phi^{op} \circ \Psi^{op} \circ F' \leq F \circ (f, \Phi)^\leftarrow \circ (g, \Psi)^\leftarrow.$$

Now from the definition of $F_{(f,\Phi)}$ it follows

$$\begin{aligned}\Psi^{op} \circ F' &\leq (\Phi^{op})^* \circ \Phi^{op} \circ \Psi^{op} \circ F' \\ &\leq (\Phi^{op})^* \circ F \circ (f, \Phi)^\leftarrow \circ (g, \Psi)^\leftarrow \\ &= F_{(f,\Phi)} \circ (g, \Psi)^\leftarrow.\end{aligned}$$

(5) Sufficiency is as in the previous case; for necessity, let $(f, \Phi) : (X, L, F) \rightarrow (Y, M, F')$ be a final morphism in $\mathbf{C} - \mathbf{FFIL}$. Since (f, Φ) is assumed to be a fuzzy filter continuous morphisms; and so by (3) we have that $F' \leq F_{(f,\Phi)}$. Using (5) and (2), and as a consequence of the finality we can conclude that

$(id_Y, id_M) : (Y, M, F') \rightarrow (Y, M, F_{(f, \Phi)})$ is a fuzzy filter continuous morphisms, which implies that

$$\begin{aligned} F_{(f, \Phi)} &= id^{op} \circ F_{(f, \Phi)} \\ &\leq F' \circ (id_Y, id_M)^{\leftarrow} \\ &= F'. \end{aligned}$$

Hence $F' = F_{(f, \Phi)}$.

This completes the proof of the proposition. ■

Theorem 2.6 (Initial fuzzy filter). *Let \mathbf{C} be a subcategory of \mathbf{LOQML} , let $(f, \Phi) : (X, L) \rightarrow (Y, M)$ be a morphism in $\mathbf{SET} \times \mathbf{C}$, where $f : X \rightarrow Y$ is an onto map, and let $F' : M^Y \rightarrow M$ be a fuzzy filter on (Y, M) , then*

$$F \equiv \Phi^{op} \circ F' \circ (f, \Phi)^{\rightarrow} : L^X \rightarrow L$$

is a fuzzy filter on (X, L) .

Proof. In fact, we have

i).

$$\begin{aligned} F(1_X) &= \Phi^{op} \left[F' (*\Phi \circ f_{L \rightarrow} (1_{(X, L)})) \right] \\ &= \Phi^{op} \left[F' (*\Phi (1_{(Y, L)})) \right] \\ &= \Phi^{op} \left[F' (1_{(Y, M)}) \right] = \Phi^{op}(\top) = \top. \end{aligned}$$

ii).

$$\begin{aligned} F(1_X) &= \Phi^{op} \left[F' (*\Phi \circ f_L^{\rightarrow} (0_{(X, L)})) \right] \\ &= \Phi^{op} \left[F' (*\Phi (0_{(Y, L)})) \right] \\ &= \Phi^{op} \left[F' (0_{(Y, M)}) \right] = \Phi^{op}(\perp) = \perp. \end{aligned}$$

iii). Let $h_1 \leq h_2$, the goal is to show that $F(h_1) \leq F(h_2)$. Since for $y \in Y$ we have

$$\begin{aligned}
(*\Phi \circ (f_L^{\rightarrow}(h_1)))(y) &= * \Phi \left(\bigvee_{f(x)=y} h_1(x) \right) \\
&= \bigwedge \left\{ m \in M \mid \bigvee_{f(x)=y} h_1(x) \leq \Phi^{op}(m) \right\} \\
&\leq \bigwedge \left\{ m \in M \mid \bigvee_{f(x)=y} h_2(x) \leq \Phi^{op}(m) \right\} \\
&= * \Phi \left(\bigvee_{f(x)=y} h_2(x) \right) = (*\Phi \circ f_L^{\rightarrow}(h_2))(y)
\end{aligned}$$

therefore $*\Phi \circ (f_L^{\rightarrow}(h_1)) \leq * \Phi \circ (f_L^{\rightarrow}(h_2))$. Hence, since F' and Φ^{op} are isotone, we have

$$\begin{aligned}
F(h_1) &= \Phi^{op} \left[F' (*\Phi \circ (f_L^{\rightarrow}(h_1))) \right] \\
&\leq \Phi^{op} \left[F' (*\Phi \circ (f_L^{\rightarrow}(h_2))) \right] \\
&= F(h_2).
\end{aligned}$$

iv). Let h and k be two elements of L^X , in order to show that $F(h) \otimes F(k) \leq F(h \otimes k)$ we proceed as follows:

Firstly, for each $y \in Y$,

$$\begin{aligned}
[f_L^{\rightarrow}(h \otimes k)](y) &= \bigvee_{f(x)=y} (h \otimes k)(x) \\
&= \bigvee_{f(x)=y} h(x) \otimes k(x) \\
&= \bigvee_{f(x)=y} h(x) \otimes \bigvee_{f(x)=y} k(x) \\
&= (f_L^{\rightarrow}(h))(y) \otimes (f_L^{\rightarrow}(k))(y);
\end{aligned}$$

consequently, $f_L^{\rightarrow}(h \otimes k) = f_L^{\rightarrow}(h) \otimes f_L^{\rightarrow}(k)$; and so

$$*\Phi \circ (f_L^{\rightarrow}(h \otimes k)) = (*\Phi \circ f_L^{\rightarrow}(h)) \otimes (*\Phi \circ f_L^{\rightarrow}(k)).$$

Finally, since F' is a fuzzy filter,

$$\begin{aligned} & F' [(*\Phi \circ f_L^{\rightarrow}(h))] \otimes F' [(*\Phi \circ f_L^{\rightarrow}(k))] \\ & \leq F' [(*\Phi \circ f_L^{\rightarrow}(h)) \otimes (*\Phi \circ f_L^{\rightarrow}(k))] \\ & = F' [* \Phi \circ (f_L^{\rightarrow}(h \otimes k))] . \end{aligned}$$

Therefore

$$\begin{aligned} F(h) \otimes F(k) &= \Phi^{op} \left[F' (*\Phi \circ f_L^{\rightarrow}(h)) \right] \otimes \Phi^{op} \left[F' (*\Phi \circ f_L^{\rightarrow}(k)) \right] \\ &= \Phi^{op} \left[F' (*\Phi \circ f_L^{\rightarrow}(h)) \otimes F' (*\Phi \circ f_L^{\rightarrow}(k)) \right] \\ &\leq \Phi^{op} \left[F' [* \Phi \circ (f_L^{\rightarrow}(h \otimes k))] \right] \\ &= F(h \otimes k) \end{aligned}$$

■

2.1. Fuzzy ultrafilters. Let $\mathfrak{F}_{FF}(X, L)$ be the set of all fuzzy filters on (X, L) . On $\mathfrak{F}_{FF}(X, L)$ we introduce a partial ordering \preceq by

$$\mathcal{F}_1 \preceq \mathcal{F}_2 \Leftrightarrow \mathcal{F}_1(f) \leq \mathcal{F}_2(f), \quad \forall (f) \in L^X$$

***Proposition 2.7.** *The partially ordered set $(\mathfrak{F}_{FF}(X, L), \preceq)$ has maximal elements.*

Proof. Referring to Zorn's lemma, it is sufficient to show that every chain \mathcal{C} in $\mathfrak{F}_{FF}(X, L)$ has an upper bound in $\mathfrak{F}_{FF}(X, L)$. For this purpose let us consider a non-empty chain $\mathcal{C} = \{\mathcal{F}_\lambda \mid \lambda \in I\}$. We define a map $\mathcal{F}_\infty : L^X \rightarrow L$ by

$$\mathcal{F}_\infty(f) = \bigvee_{\lambda \in I} \mathcal{F}_\lambda(f),$$

and we show that \mathcal{F}_∞ is a fuzzy filter on (X, L) . In fact

$$(FF1.b.i) \quad \mathcal{F}_\infty(1_X) = \bigvee_{\lambda \in I} \mathcal{F}_\lambda(1_X) = \bigvee_{\lambda \in I} \top = \top.$$

$$(FF1.b.ii) \quad \mathcal{F}_\infty(0_X) = \bigvee_{\lambda \in I} \mathcal{F}_\lambda(0_X) = \bigvee_{\lambda \in I} \perp = \perp.$$

$$(FF1.b.iii) \quad f \leq g \Rightarrow \mathcal{F}_\infty(f) = \bigvee_{\lambda \in I} \mathcal{F}_\lambda(f) \leq \bigvee_{\lambda \in I} \mathcal{F}_\lambda(g) = \mathcal{F}_\infty(g).$$

$$(FF1.b.iv)$$

$$\begin{aligned} \mathcal{F}_\infty(f) \otimes \mathcal{F}_\infty(g) &= \left(\bigvee_{\lambda \in I} \mathcal{F}_\lambda(f) \right) \otimes \left(\bigvee_{\lambda \in I} \mathcal{F}_\lambda(g) \right) \\ &= \bigvee_{\lambda \in I} [\mathcal{F}_\lambda(f) \otimes \mathcal{F}_\lambda(g)] \\ &\leq \bigvee_{\lambda \in I} [\mathcal{F}_\lambda(f \otimes g)] \\ &= \mathcal{F}_\infty(f \otimes g). \end{aligned}$$

■

Definition 2.8. A maximal element in $(\mathfrak{F}_{FF}(X), \preceq)$ is also called a fuzzy ultrafilter.

***Proposition 2.9.** For every fuzzy filter $\mathcal{U} : L^X \rightarrow L$ on X the following assertions are equivalent

- (i) \mathcal{U} is a fuzzy ultrafilter.
- (ii) $\mathcal{U}(f) = [\mathcal{U}(f \rightarrow 0_X)] \rightarrow \perp$, for all $f \in L^X$.

Proof. (i) \Rightarrow (ii)

Because of (FF1.b.iii) and (FF1.b.iv) every fuzzy filter satisfies the condition

$$(FF1.b.ii') \quad \mathcal{U}(f) \leq [\mathcal{U}(f \rightarrow 0_X)] \rightarrow \perp, \text{ for all } f \in L^X.$$

In order to verify (i) \Rightarrow (ii) it is sufficient to show that the maximality of \mathcal{U} implies

$$[\mathcal{U}(f \rightarrow 0_X)] \rightarrow \perp \leq \mathcal{U}(f), \quad \forall f \in L^X.$$

For this purpose, we fix an element $g \in L^X$, for that element we let $\mathcal{G}_g := [\mathcal{U}(g \rightarrow 0_X)] \rightarrow \perp$ and define a map $\hat{\mathcal{U}} : L^X \rightarrow L$ by

$$\hat{\mathcal{U}}(f) = \mathcal{U}(f) \bigvee \left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\}.$$

We must show that $\hat{\mathcal{U}}$ is a fuzzy ultrafilter. Firstly $\hat{\mathcal{U}}$ is a fuzzy filter: obviously $\hat{\mathcal{U}}$ satisfies (FF1.b.i).

In order to verify (FF1.b.ii), we have that

$$\begin{aligned} \hat{\mathcal{U}}(0_X) &= \mathcal{U}(0_X) \bigvee \left\{ \mathcal{U}(g \rightarrow 0_X) \otimes \mathcal{G}_g \right\} \\ &= \perp \vee \left\{ \mathcal{U}(g \rightarrow 0_X) \otimes \mathcal{G}_g \right\} \\ &= \mathcal{U}(g \rightarrow 0_X) \otimes \mathcal{G}_g. \end{aligned}$$

Now we invoke the residuation property of (L, \leq, \otimes) to obtain

$$\hat{\mathcal{U}}(0_X) = \left\{ \mathcal{U}(g \rightarrow 0_X) \otimes \mathcal{G}_g \right\} = \perp.$$

For the axiom (FF1.b.iii), from the definition

$$\hat{\mathcal{U}}(f) = \mathcal{U}(f) \bigvee \left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\}$$

and

$$\hat{\mathcal{U}}(h) = \mathcal{U}(h) \bigvee \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\}.$$

Now, for $f \leq h$ we have that $\mathcal{U}(f) \leq \mathcal{U}(h)$, moreover,

$$\begin{aligned} g \rightarrow f &= \bigvee \{ k \in L^X \mid g \otimes k \preceq f \} \\ &\leq \bigvee \{ k \in L^X \mid g \otimes k \preceq h \} \\ &= g \rightarrow h, \end{aligned}$$

which implies that

$$\begin{aligned}\hat{\mathcal{U}}(f) &= \mathcal{U}(f) \bigvee \left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\} \\ &\leq \mathcal{U}(h) \bigvee \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\} = \hat{\mathcal{U}}(h).\end{aligned}$$

For the axiom (*FF1.b.iv*), we must verify that

$$\hat{\mathcal{U}}(f) \otimes \hat{\mathcal{U}}(h) \leq \hat{\mathcal{U}}(f \otimes h),$$

In fact,

$$\begin{aligned}\hat{\mathcal{U}}(f) \otimes \hat{\mathcal{U}}(h) &= (\mathcal{U}(f) \bigvee \left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\}) \otimes (\mathcal{U}(h) \bigvee \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\}) \\ &= \mathcal{U}(f) \otimes \left[\mathcal{U}(h) \bigvee \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\} \right] \\ &\quad \bigvee \left[\left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\} \otimes \left\{ \mathcal{U}(h) \bigvee \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\} \right\} \right] \\ &= \mathcal{U}(f) \otimes \mathcal{U}(h) \bigvee \left[\mathcal{U}(f) \otimes \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\} \right] \\ &\quad \bigvee \left(\left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\} \otimes \mathcal{U}(h) \right) \bigvee \left(\left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\} \otimes \left\{ \mathcal{U}(g \rightarrow h) \otimes \mathcal{G}_g \right\} \right) \\ &= \mathcal{U}(f) \otimes \mathcal{U}(h) \bigvee \left[\left\{ \mathcal{U}(f) \otimes \mathcal{U}(g \rightarrow h) \right\} \otimes \mathcal{G}_g \right] \\ &\quad \bigvee \left(\left\{ \mathcal{U}(h) \otimes \mathcal{U}(g \rightarrow f) \right\} \otimes \mathcal{G}_g \right) \bigvee \left(\left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{U}(g \rightarrow h) \right\} \otimes \mathcal{G}_g \right) \\ &\leq \mathcal{U}(f \otimes h) \bigvee \left[\mathcal{U}(f \otimes [g \rightarrow h]) \otimes \mathcal{G}_g \right] \\ &\quad \bigvee \left(\mathcal{U}(h \otimes [g \rightarrow f]) \otimes \mathcal{G}_g \right) \bigvee \left(\mathcal{U}((g \rightarrow f) \otimes [g \rightarrow h]) \otimes \mathcal{G}_g \right) \\ &\leq \mathcal{U}(f \otimes h) \bigvee \left[\mathcal{U}(g \rightarrow f \otimes h) \otimes \mathcal{G}_g \right] \\ &\quad \bigvee \left[\mathcal{U}(g \rightarrow f \otimes h) \otimes \mathcal{G}_g \right] \bigvee \left[\mathcal{U}(g \rightarrow f \otimes h) \otimes \mathcal{G}_g \right] \\ &= \mathcal{U}(f \otimes h) \bigvee \left[\mathcal{U}(g \rightarrow f \otimes h) \otimes \mathcal{G}_g \right] \\ &= \hat{\mathcal{U}}(f \otimes h).\end{aligned}$$

Now we must show that $\hat{\mathcal{U}}$ is a fuzzy ultrafilter on X . In fact, since

$$\hat{\mathcal{U}}(f) = \mathcal{U}(f) \bigvee \left\{ \mathcal{U}(g \rightarrow f) \otimes \mathcal{G}_g \right\},$$

clearly $\mathcal{U}(f) \leq \hat{\mathcal{U}}(f)$, $\forall f \in L^X$, but \mathcal{U} is a fuzzy ultrafilter on X , therefore $\hat{\mathcal{U}} = \mathcal{U}$. In this way

$$\begin{aligned}\mathcal{U}(g) &= \mathcal{U}(g) \bigvee \left\{ \mathcal{U}(g \rightarrow g) \otimes \mathcal{G}_g \right\} \\ &= \mathcal{U}(g) \bigvee \left\{ \mathcal{U}(1_X) \otimes \mathcal{G}_g \right\} \\ &= \mathcal{U}(g) \bigvee \left\{ \top \otimes \mathcal{G}_g \right\} \\ &= \mathcal{U}(g) \vee \mathcal{G}_g.\end{aligned}$$

Therefore,

$$\mathcal{G}_g = [\mathcal{U}(g \rightarrow 0_X)] \rightarrow \perp \leq \mathcal{U}(g), \quad \forall g \in L^X.$$

From the last inequality and (FF1.b.ii') we obtain (ii).

(ii) \Rightarrow (i)

We must verify that if

$$\mathcal{U}(f) = (\mathcal{U}(f \rightarrow 0_X)) \rightarrow \perp, \quad \text{for all } f \in L^X,$$

then \mathcal{U} is a fuzzy ultrafilter on X .

Suppose $\mathcal{U} \leq \hat{\mathcal{U}}$, then

$$\left([\hat{\mathcal{U}}(f \rightarrow 0_X)] \rightarrow \perp \right) \leq ([\mathcal{U}(f \rightarrow 0_X)] \rightarrow \perp),$$

therefore $\hat{\mathcal{U}} \leq \mathcal{U}$, consequently \mathcal{U} is an L -fuzzy ultrafilter on X . ■

***Proposition 2.10.** *Let $\phi : X \rightarrow Y$ be a map and let $\mathcal{F} : L^X \rightarrow L$ be a fuzzy filter on X . Then the map $\phi_{\mathcal{U}}^{\rightarrow} : L^Y \rightarrow L$ is a fuzzy ultrafilter on Y , whenever \mathcal{U} will be a fuzzy ultrafilter on X*

Proof. Let $\mathcal{U} : L^X \rightarrow L$ be a fuzzy ultrafilter on X and let $g \in L^Y$, then

$$\begin{aligned}
\phi_{\mathcal{U}}^{\rightarrow}(g) &= \mathcal{U}(g \circ \phi) \\
&= \mathcal{U}((g \circ \phi) \rightarrow 0_X) \rightarrow \perp \\
&= \mathcal{U}[(g \circ \phi) \rightarrow (0_Y \circ \phi)] \rightarrow \perp \\
&= \mathcal{U}[(g \rightarrow 0_Y) \circ \phi] \rightarrow \perp \\
&= \phi_{\mathcal{U}}^{\rightarrow}[g \rightarrow 0_Y] \rightarrow \perp.
\end{aligned}$$

We conclude from proposition 2.9 that $\phi_{\mathcal{U}}^{\rightarrow} : L^Y \times L \rightarrow L$ is an L -fuzzy ultrafilter on Y . ■

3. THE CATEGORY $\mathbf{C} - \mathbf{FTOP}$

In this section we transcribe some facts about categorical topology, taken from [6], in order to establish (in the next section) a relationship between this category and the category $\mathbf{C} - \mathbf{FFIL}$.

Definition 3.1 (The category $\mathbf{C} - \mathbf{FTOP}$). *Let \mathbf{C} be a subcategory of \mathbf{LOQML} . The category $\mathbf{C} - \mathbf{FTOP}$ comprises the following data:*

(CF1) **Objects.** *Objects are ordered triples (X, L, Υ) satisfying the following axioms:*

- (a) **Ground axiom.** $(X, L) \in |\mathbf{SET} \times \mathbf{C}|$,
- (b) **Fuzzy topological axiom.** $\Upsilon : L^X \rightarrow L$ is a mapping satisfying:

- (i) *For all set of index J , para todo $\{f_{\lambda} \mid \lambda \in J\} \subseteq L^X$,*

$$\bigwedge_{\lambda \in J} \Upsilon(f_{\lambda}) \leq \Upsilon\left(\bigvee_{\lambda \in J} f_{\lambda}\right)$$

- (ii) $\Upsilon(f) \otimes \Upsilon(g) \leq \Upsilon(f \otimes g), \quad \forall f, g \in L^X$

- (iii) $\Upsilon(1_X) = \top$.

(c) **Equality of objects.** $(X, L, \Upsilon) = (Y, M, \Gamma)$ iff $(X, L) = (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$ and $\Upsilon = \Gamma$ as SET mappings from $L^X \equiv M^Y$ to $L \equiv M$.

(CF2) **Morphisms.** Morphisms are ordered pairs

$$(f, \Phi) : (X, L, \Upsilon) \rightarrow (Y, M, \Gamma)$$

called **fuzzy continuous morphisms**, satisfying the following axioms:

- (a) **Ground axiom.** $(f, \Phi) : (X, L) \rightarrow (Y, M)$ is a morphism in $\mathbf{SET} \times \mathbf{C}$
- (b) **fuzzy continuity axiom** $\Phi^{op} \circ \Gamma \leq \Upsilon \circ (f, \Phi)^\leftarrow$ on M^Y .
- (c) **Equality of morphisms.** As in $\mathbf{SET} \times \mathbf{C}$.

(CF3) **Composition.** As in $\mathbf{SET} \times \mathbf{C}$.

(CF4) **Identities.** As in $\mathbf{SET} \times \mathbf{C}$.

The ordered triple (X, L, Υ) is a **fuzzy topological space** on the ground set (X, L) .

Proposition 3.2 (Alternate fuzzy continuity axiom). [6] On M^Y the following holds:

$$\Phi^{op} \circ \Gamma \leq \Upsilon \circ (f, \Phi)^\leftarrow \quad \text{if and only if} \quad \Gamma \leq (\Phi^{op})^* \circ \Upsilon \circ (f, \Phi)^\leftarrow,$$

where $(\Phi^{op})^*$ is the right adjoint of Φ^{op} .

Theorem 3.3 (Final structures and morphisms for fuzzy topology). [6] Let \mathbf{C} be a subcategory of \mathbf{LOQML} , let $(f, \Phi) : (X, L) \rightarrow (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$, let $\Phi^* \equiv (\Phi^{op})^*$ be the right adjoint of $\Phi^{op} : L \leftarrow M$ and let Υ be a fuzzy topology on (X, L) . Then the following holds:

- (1) $\Upsilon_{(f, \Phi)} \equiv \Phi^* \circ \Upsilon \circ (f, \Phi)^\leftarrow : Y \rightarrow M$ is a fuzzy topology on (Y, M) ;
- (2) $(f, \Phi) : (X, L, \Upsilon) \rightarrow (Y, M, \Upsilon_{(f, \Phi)})$ is fuzzy continuous;

- (3) $(f, \Phi) : (X, L, \Upsilon) \rightarrow (Y, M, \Upsilon')$ is fuzzy continuous iff $\Upsilon' \leq \Upsilon_{(f, \Phi)}$;
- (4) $\Upsilon_{(f, \Phi)}$ is the join of all the fuzzy topologies Υ' on M^Y for which $(f, \Phi) : (X, L, \Upsilon) \rightarrow (Y, M, \Upsilon')$ is fuzzy continuous;
- (5) $(f, \Phi) : (X, L, \Upsilon) \rightarrow (Y, M, \Upsilon_{(f, \Phi)})$ is a final morphism in $\mathbf{C} - \mathbf{FTOP}$;
- (6) $(f, \Phi) : (X, L, \Upsilon) \rightarrow (Y, M, \Upsilon')$ is a final morphism in $\mathbf{C} - \mathbf{FTOP}$ iff $\Upsilon' \leq \Upsilon_{(f, \Phi)}$.

Theorem 3.4 ($\mathbf{C} - \mathbf{FTOP}$ is a topological category). [6] *For each subcategory \mathbf{C} of \mathbf{LOQML} , the category $\mathbf{C} - \mathbf{FTOP}$ is topological over $\mathbf{SET} \times \mathbf{C}$ with respect to the forgetful functor V .*

4. FROM $\mathbf{C} - \mathbf{FFIL}$ TO $\mathbf{C} - \mathbf{FTOP}$

There exists a natural relationship between the categories $\mathbf{C} - \mathbf{FFIL}$ and $\mathbf{C} - \mathbf{FTOP}$. Our purpose in this section is to describe it, as a generalization of [5].

If we compare the axiom (CF1.b) from definition 3.1 of fuzzy topology with axiom (FF1.b) from definition 2.1 of fuzzy filter, we can see that the condition (ii) of the latter implies condition (i) of the former, in fact:

Let $J \neq \emptyset$ be an index set and let $\{f_\lambda \mid \lambda \in J\} \subseteq L^X$, then we have

$$f_\lambda \leq \bigvee_{\lambda \in J} f_\lambda \quad \text{for each } \lambda \in J;$$

invoking (CF1.b.iii), we get

$$F(f_\lambda) \leq F\left(\bigvee_{\lambda \in J} f_\lambda\right) \quad \text{for each } \lambda \in J,$$

therefore

$$\bigwedge_{\lambda \in J} F(f_\lambda) \leq F\left(\bigvee_{\lambda \in J} f_\lambda\right).$$

Moreover, if we change (FF1.b.ii) from the definition of fuzzy filter by $F(0_X) = \top$, it is obtained.

Proposition 4.1 (Fuzzy filtered-type topolgy). *Let $F : L^X \rightarrow L$ be a fuzzy filter on (X, L) . Then the mapping $\Upsilon_F : L^X \rightarrow L$ defined , for each $g \in L^X$, by*

$$\Upsilon_F(g) = \begin{cases} F(g) & \text{if } g \neq 0_X \\ \top & \text{if } g = 0_X \end{cases}$$

is a fuzzy topology on (X, L) .

***Corollary 4.2** (Fuzzy ultra-filtered-type topolgy). *Let $\mathcal{U} : L^X \rightarrow L$ be a fuzzy ultrafilter on (X, L) . Then the mapping $\Upsilon_{\mathcal{U}} : L^X \rightarrow L$ defined , for each $g \in L^X$, by*

$$\Upsilon_{\mathcal{U}}(g) = \begin{cases} \mathcal{U}(g) & \text{if } g \neq 0_X \\ \top & \text{if } g = 0_X \end{cases}$$

is a fuzzy ultra-topology on (X, L) .

Thus, if we have a fuzzy filtered set (X, L, F) , we obtain the fuzzy topological space (X, L, \mathcal{T}_F) ; moreover, if ϕ is a morphism between (X, L, F) and (Y, M, G) in $\mathbf{C} - \mathbf{FFIL}$, the *same mappping* ϕ is also a morphism in $\mathbf{C} - \mathbf{FTOP}$, and the diagram

$$\begin{array}{ccc} (X, F) & \longrightarrow & (X, \mathcal{T}_F) \\ \phi \downarrow & & \downarrow \phi_* = \phi \\ (Y, G) & \longrightarrow & (Y, \mathcal{T}_G) \end{array}$$

is commutative. Also, we observe that if $\phi \neq \psi$ are morphisms in $\mathbf{C} - \mathbf{FFIL}$ then $\phi_* \neq \psi_*$. In other words,

Theorem 4.3. *The funtion $\mathcal{T} : \mathbf{C} - \mathbf{FFIL} \rightarrow \mathbf{C} - \mathbf{FTOP}$ that assigns to each object (X, L, F) of $\mathbf{C} - \mathbf{FFIL}$ the object (X, L, \mathcal{T}_F) of $\mathbf{C} - \mathbf{FTOP}$, and to each morphism ϕ in $\mathbf{C} - \mathbf{FFIL}$ the morphism $\phi_* = \phi$ in $\mathbf{C} - \mathbf{FTOP}$ is a faithful functor between the category $\mathbf{C} - \mathbf{FFIL}$ of fuzzy filtered sets, and the category $\mathbf{C} - \mathbf{FTOP}$ of fuzzy topological spaces.*

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